Numerical modelling of large amplitude long waves in the coastal zone using COULWAVE code

Tomas Torsvik

Bergen Center for Computational Science
UNIFOB
University of Bergen

Guadeloupe
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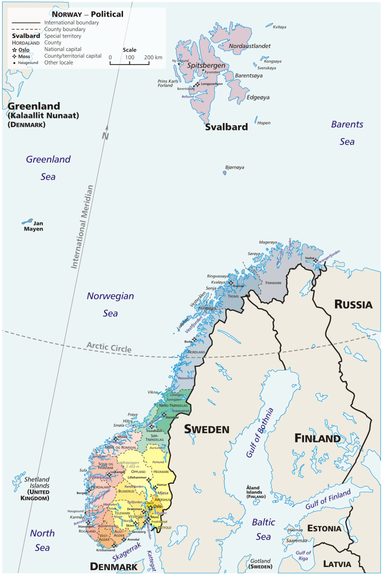
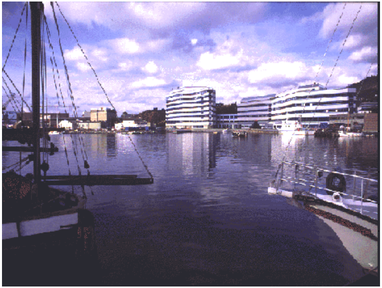




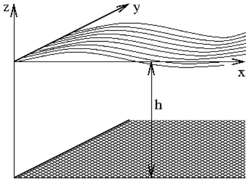
Overview

- Basic equations
  - Nonlinear equations
  - Dispersive equations
- Properties of the COULWAVE model
  - Boundary conditions:
    - Open boundaries
    - Wetting/drying along the coast line
  - Wave breaking
  - Bottom friction
- Examples
  - Breaking and non-breaking wave runup
  - Simulation using realistic wave fields and bathymetry
  - Ship waves

Equations for inviscid, incompressible flow

- Continuity equation:

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$



- Euler equation:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g}$$

- Dynamic and kinematic boundary conditions:

$$p = p_a(x, y, t), \quad \text{at} \quad z = \eta(x, y, t)$$

$$w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}, \quad \text{at} \quad z = \eta(x, y, t)$$

$$w = u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}, \quad \text{at} \quad z = -h(x, y)$$

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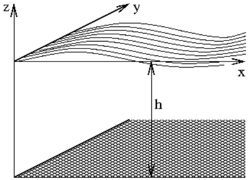

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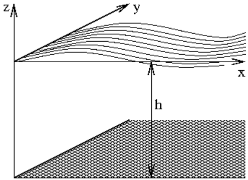

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Linear wave theory

- Use linearized equations and assume a wave solution:

$$\eta = \eta_{max} \sin(kx - \omega t)$$

- Dispersion relation for linear wave:

$$\omega^2 = gk \tanh(kh)$$

- Phase velocity:

$$c = \frac{\omega}{k} = \left(\frac{g}{k} \tanh(kh)\right)^{1/2}$$

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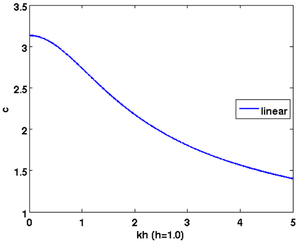

Shallow water approximation

- For $kh \rightarrow 0$
  - Dispersion relation:

$$\omega^2 \approx gk^2 h$$

  - Phase speed:

$$c = \sqrt{gh}$$

- Long waves are non-dispersive.
- The wave energy is transported with the individual waves, i.e. there is little or no exchange of wave energy between waves in a wave train propagating in shallow water.

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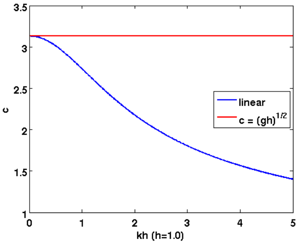

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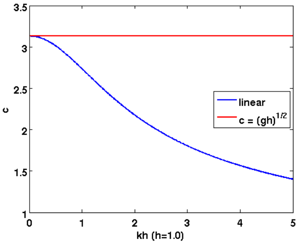

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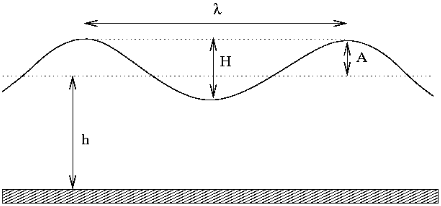
Scales for surface gravity waves



Acceleration scale:
g - acceleration of gravity

Length scales:
λ - wave length, h - depth, A - amplitude/elevation

Time scale:
Usually constructed from length and acceleration scales, e.g.
$t = \lambda/\sqrt{gh}$

Approximations for shallow water flow

Shallow water, long wave theory: $\epsilon = \frac{h}{\lambda} \ll 1$

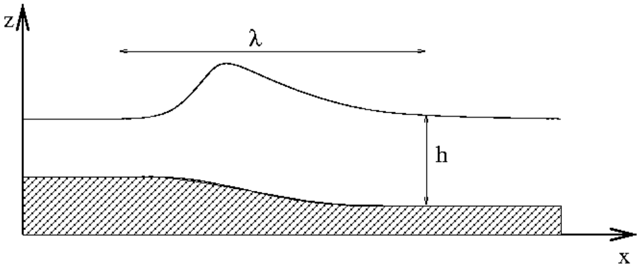
Weakly non-linear theory: $\mu = \frac{A}{h}, \frac{A}{\lambda} \ll 1$

Ursell number: $Ur = \frac{\epsilon}{\mu^2} = \frac{A\lambda^2}{h^3}$

- Linear theory: $Ur \ll 1$
- Weakly dispersive, weakly non-linear waves: $Ur \approx 1$
- Non-dispersive, finite amplitude waves: $Ur \gg 1$

Comparison of length scales indicates if waves are non-linear or dispersive, or if both effects should be included.

Shallow water theory



2D case discussed for simplicity
U, W - characteristic horizontal and vertical velocities

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \Rightarrow \frac{W}{h} \sim \frac{U}{\lambda} \Rightarrow \frac{W}{U} \sim \frac{h}{\lambda} \ll 1$$

Strategies for numerical simulations

Numerical solution of Euler equations is possible, but at high computational cost.
- Restrict computational domain (or resolution)

Shallow water approximation:
- Reduce 3D equations to 2D equations by integrating over depth.
- Derive governing equations by expanding $\mathbf{u}$ in terms of z.
Several different formulations may be derived from the primitive equations.

Korteweg-de Vries equation

Simplest form of weakly nonlinear, weakly dispersive equation:

$$\frac{\partial \eta}{\partial t} + \left(1 + \frac{3}{2}\eta\right)\frac{\partial \eta}{\partial x} + \frac{1}{6}\frac{\partial^3 \eta}{\partial x^3} = 0$$

Compare with properties for the nonlinear transport equation

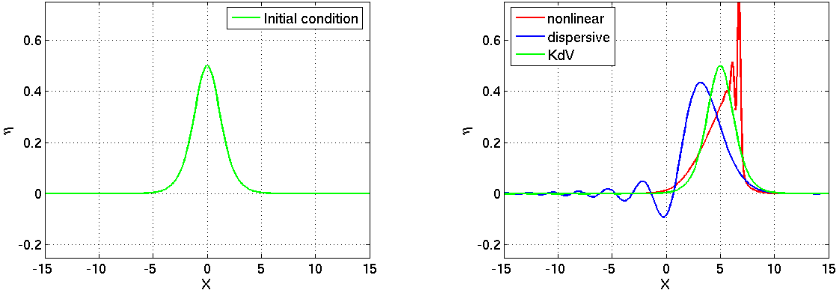
$$\frac{\partial \eta}{\partial t} + \left(1 + \frac{3}{2}\eta\right)\frac{\partial \eta}{\partial x} = 0$$

and the linear weakly dispersive equation

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} + \frac{1}{6}\frac{\partial^3 \eta}{\partial x^3} = 0$$

Korteweg-de Vries equation (Cont.)

Evolution of an initial disturbance $\eta = A\cosh^{-2}\left[\frac{\sqrt{3A}}{2}x\right]$



fKdV and KP equations

Forced KdV (fKdV) equation (1D propagation)

$$\frac{\partial \eta}{\partial t} + \left(1 + \frac{3}{2}\eta\right)\frac{\partial \eta}{\partial t} + \frac{1}{6}\frac{\partial^3 \eta}{\partial x^3} = \frac{1}{2}\frac{\partial p}{\partial x}$$

Kadomtsev-Petviashvili (KP) equation (2D propagation)

$$\frac{\partial}{\partial x}\left[\frac{\partial \eta}{\partial t} + \left(1 + \frac{3}{2}\eta\right)\frac{\partial \eta}{\partial t} + \frac{1}{6}\frac{\partial^3 \eta}{\partial x^3}\right] - \frac{1}{2}\frac{\partial^2 \eta}{\partial y^2} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}$$

Derived from KdV equation by relaxing 1D requirement.

Boussinesq equations

- Formulate equations in terms of depth averaged velocity $\bar{\mathbf{u}}$, where $\nabla_H = (\partial/\partial x, \partial/\partial y)$.
- Continuity equation:

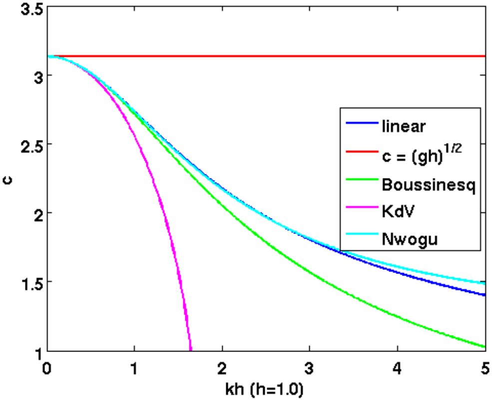
$$\frac{\partial \eta}{\partial t} + \nabla_H \cdot [(h + \eta)\bar{\mathbf{u}}] = 0$$

- Momentum equation:

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla_H)\bar{\mathbf{u}} = -\nabla_H \eta - \nabla_H p_a + \frac{1}{3} h^2 \nabla_H \nabla_H \cdot \left(\frac{\partial \bar{\mathbf{u}}}{\partial t} \right)$$

- Formulations with depth variation include $\nabla_H h$ terms.
- Higher order formulations include dispersive terms with nonlinear corrections.

Dispersion relation for long wave equations



Linear: $c = \sqrt{(g/k)\tanh(kh)}$
Boussinesq: Classical Boussinesq formulation
Nwogu: Improved Boussinesq formulation

Properties of long wave equations

KdV and KP equations:

+ Closed form (explicit) solutions exist for solitary and periodic waves.
- Restricted to unidirectional (KdV) or narrow angle (KP) of wave propagation.
- Poor dispersion relation for intermediate water depth.

Boussinesq equations:

- No known closed form solutions.
+ No preferred direction of wave propagation.
+ Improved formulation with reasonable dispersion relation up to $kh \approx \pi$.

The COULWAVE model

- COULWAVE: Cornell University Long Wave model
  `http://ceprofs.tamu.edu/plynett/COULWAVE/`
- Boussinesq type model equations
- High order predictor-corrector method (Adams-Bashforth + Adams-Moulton) for time stepping
- Spatial derivatives approximated by finite differences
- Equations discretized on a uniform, quadrilateral grid
- Depth a function of space and time $h = h(\mathbf{x}, t)$, which allows simulation of underwater landslide

- FUNWAVE: Long wave model similar to COULWAVE
  `http://chinacat.coastal.udel.edu/~kirby/programs/funwave/funwave.html`

Continuity equation

$$
\begin{aligned}
&\frac{1}{\epsilon}\frac{\partial h}{\partial t} + \frac{\partial \eta}{\partial t} + \nabla \cdot [(h + \epsilon\eta)\bar{\mathbf{u}}] \\
&- \mu^2 \nabla \cdot \left\{ \left[\frac{h^3 + \epsilon^3\eta^3}{6} - \frac{(h+\epsilon\eta) z_\alpha^2}{2} \right] \nabla S \right. \\
&- \left. \left[\frac{h^2 - \epsilon^2\eta^2}{2} + (h+\epsilon\eta) z_\alpha \right] \nabla T \right\} = \mathcal{O}(\mu^4)
\end{aligned}
\tag{1}
$$

where

$$
S = \nabla \cdot \bar{\mathbf{u}}\,, \qquad T = \nabla \cdot (h\bar{\mathbf{u}}) + \frac{1}{\epsilon}\frac{\partial h}{\partial t}
$$

and z_α is a reference depth, usually $z_\alpha = -0.531h$

Momentum equation

$$
\frac{\partial \bar{\mathbf{u}}}{\partial t} + \epsilon \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} + \nabla \eta + \nabla p_a + \mu^2 \frac{\partial}{\partial t} \left\{ \frac{z_\alpha^2}{2} \nabla S + z_\alpha \nabla T \right\}
$$

$$
+ \epsilon \mu^2 \left[(\bar{\mathbf{u}} \cdot \nabla z_\alpha)(z_\alpha \nabla S + \nabla T) + z_\alpha \nabla (\bar{\mathbf{u}} \cdot \nabla T) + \frac{z_\alpha^2}{2} \nabla (\bar{\mathbf{u}} \cdot \nabla S) \right]
$$

$$
+ \epsilon \mu^2 \left[T \nabla T - \nabla \left(\eta \frac{\partial T}{\partial t} \right) \right]
$$

$$
+ \epsilon^2 \mu^2 \nabla \left(\eta S T - \frac{\eta^2}{2} \frac{\partial S}{\partial t} - \eta \bar{\mathbf{u}} \cdot \nabla T \right)
$$

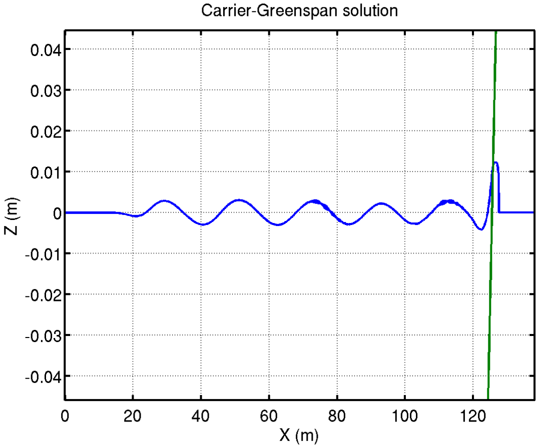
$$
+ \epsilon^3 \mu^2 \nabla \left[\frac{\eta^2}{2} \left(S^2 - \bar{\mathbf{u}} \cdot \nabla S \right) \right] = \mathcal{O}(\mu^4) \tag{2}
$$

Boundary conditions

- Open boundaries: Flow Relaxation Zones (Sponge Layer)
  - Multiply $\bar{\mathbf{u}}$ by exponentially decaying function near open boundaries.
- Closed boundaries: Full reflection of waves approaching the boundary
- Wet-dry boundary: Moving boundary at the shore line
  - Linear extrapolation of water phase into dry region

Example: Carrier-Greenspan solution

- Carrier-Greenspan solution for runup and rundown on a plane beach
- Wave height 0.006 m, period 10 s, depth 0.5 m, slope 1:25



Wave breaking

- Wave breaking model after Kennedy et al. (2000)[1]
- Add an eddy viscosity term to the momentum equation.

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \epsilon \bar{\mathbf{u}} \cdot \nabla_H \bar{\mathbf{u}} + \nabla_H \eta + \nabla_H p_a - \frac{1}{3}\mu^2 h^2 \nabla_H \nabla_H \cdot \left(\frac{\partial \bar{\mathbf{u}}}{\partial t} \right) - \mathbf{R}_b(\nu) = \mathcal{O}(\mu^4)$$

- Eddy viscosity: $\nu = B\delta_b^2 (h + \eta)\eta_t$
  δ_b - nondimensional mixing length, usually $0.9 < \delta_b < 1.5$

[1] J. Waterway, Port, Coastal, and Ocean Eng. (39), 2000

Wave breaking (cont.)

- Breaking occurs when $\eta_t > \eta_t^*$

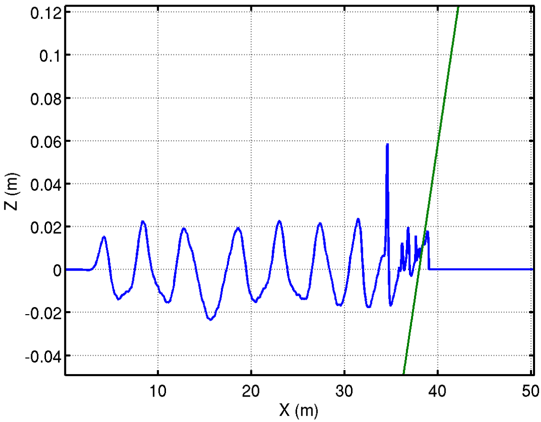
$$B = \begin{cases} 1\,, & \eta_t \geq 2\eta_t^* \\ \eta_t/\eta_t^* - 1\,, & \eta_t^* \leq \eta_t < 2\eta_t^* \\ 0\,, & \eta_t < \eta_t^* \end{cases}$$

- Once triggered, η_t^* decreases with time from $\eta_t^{*(I)}$ to $\eta_t^{*(F)}$

$$\eta_t^* = \begin{cases} \eta_t^{*(I)} + \dfrac{t - t_0}{T^*}(\eta_t^{*(F)} - \eta_t^{*(I)})\,, & 0 \leq (t - t_0) < T^* \\ \eta_t^{*(F)}\,, & (t - t_0) \geq T^* \end{cases}$$

Example: Waves breaking on a plane beach

- Regular waves breaking on a beach
- Wave height 0.036 m, depth 0.36 m, slope 1:34.26



Bottom friction

- Bottom friction formulations:
  - Standard quadratic formulation

$$R = C_{FR}\bar{\mathbf{u}}|\bar{\mathbf{u}}|$$

  - Convolution integral formulation

$$R = C_{FR}\int_0^t \frac{\nabla_H \cdot \bar{\mathbf{u}}}{\sqrt{t-\tau}}\, d\tau$$

Bottom friction: Standard formulation

- Boussinesq equations with standard bottom friction formulation
  Continuity equation:

$$\frac{\partial \eta}{\partial t} + \nabla_H \cdot [(h + \epsilon\eta)\bar{\mathbf{u}}] = \mathcal{O}(\mu^4)$$

  Momentum equation:

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \epsilon\bar{\mathbf{u}} \cdot \nabla_H \bar{\mathbf{u}} + \nabla_H \eta + \nabla_H p_a$$
$$-\frac{1}{3}\mu^2 h^2 \nabla_H \nabla_H \cdot \left(\frac{\partial \bar{\mathbf{u}}}{\partial t}\right) + C_{FR}\bar{\mathbf{u}}|\bar{\mathbf{u}}| = \mathcal{O}(\mu^4)$$

Bottom friction: Convolution integral formulation

- Viscous effects can be important near solid boundaries.
- Boussinesq equations with a bottom friction derived from boundary layer analysis (Liu and Orfila[2]):

$$\frac{\partial \eta}{\partial t} + \nabla_H \cdot [(h + \epsilon\eta)\bar{\mathbf{u}}] - \frac{\delta}{\mu\sqrt{\pi}} \int_0^t \frac{\nabla_H \cdot \bar{\mathbf{u}}}{\sqrt{t-\tau}}\, d\tau = \mathcal{O}(\mu^4)$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \epsilon\bar{\mathbf{u}} \cdot \nabla_H \bar{\mathbf{u}} + \nabla_H \eta + \nabla_H p_a - \frac{1}{3}\mu^2 h^2 \nabla_H \nabla_H \cdot \left(\frac{\partial \bar{\mathbf{u}}}{\partial t}\right) = \mathcal{O}(\mu^4)$$

$$\delta^2 = \frac{\nu}{\lambda\sqrt{gh}}, \qquad \mathcal{O}(\delta) \approx \mathcal{O}(\mu^4) \approx \mathcal{O}(\epsilon^2)$$

[2] J. Fluid Mech.(2004), vol. 520

Approximation of convolution integral

- Large computational cost for computing the exact convolution integral.
- We can estimate the value of the convolution integral by

$$\int_{t-t^*}^{t} \frac{\nabla_H \cdot \bar{\mathbf{u}}}{\sqrt{t-\tau}}\, d\tau + C_R R(t-t^*)$$

  where t^* is the truncation time, $R(t-t^*)$ is a time dependent residual term, and C_R is the residual coefficient.

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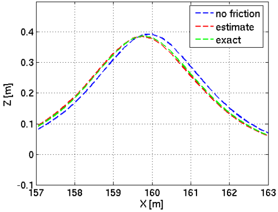
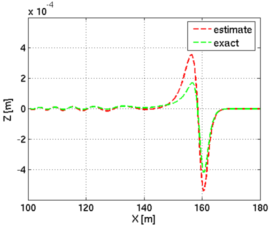
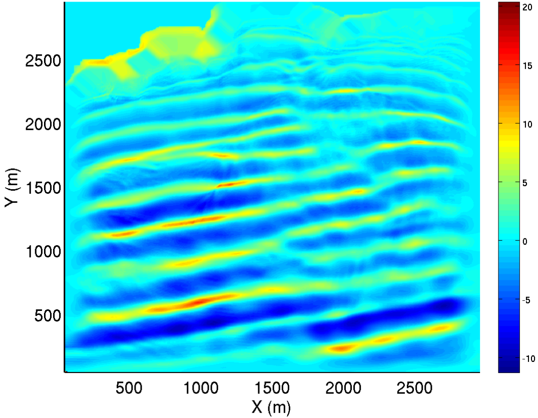




Example: Complex topographies

- Simulating irregular wave train in Wiamea Bay, Hawaii



Example: Wash waves from high speed vessels

HSS "Stena Discovery"

Top Speed:

40 knots (= 20.6 m/s)

Dimensions:

Length:	121.75 m
Width:	40.00 m
Draft:	4.80 m

- Waves from high speed vessels:
  - Long wave lengths and wave periods.
  - Large wave energy.
  - Qualitatively different from waves generated by conventional ships.
- Potentially dangerous for people on the shore or in small boats.
- May damage structures at the shore or moored vessels.
- May increase erosion and disturb marine habitats.

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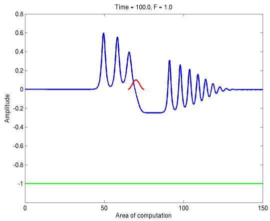
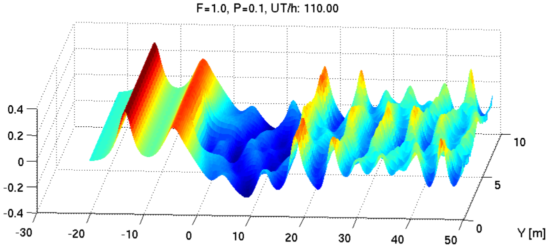
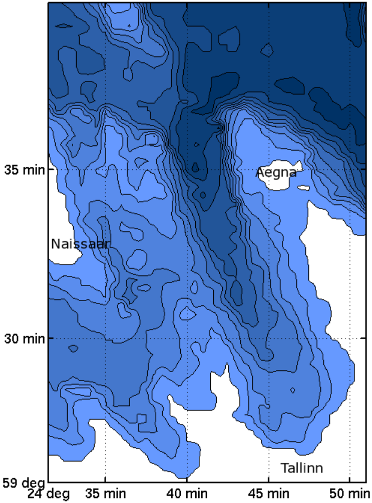
Waves generated by a pressure disturbance

- Pressure included through dynamic boundary condition. Consistent with primitive equations.
- Simple to implement.
- Difficult to represent specific hull shapes.





Ship waves in Tallinn Bay



- Basin of about 10 km x 20 km
- Ship traffic follows NW - SE underwater valley, with depth ranging from 10 m to 90 m.
- Daily crossings (ingoing or outgoing)
    - 22 HSC/Catamaran
    - 8-10 Hydrofoil
- A natural laboratory for the study of long waves.
    - Also relevant for tsunami research.

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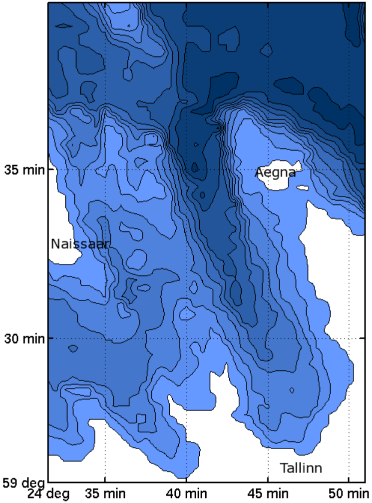

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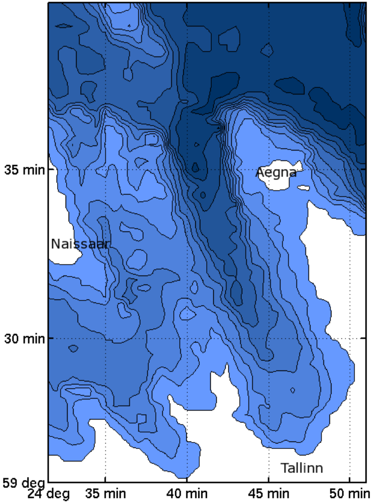
Ship waves in Tallinn Bay



- Basin of about 10 km x 20 km
- Ship traffic follows NW - SE underwater valley, with depth ranging from 10 m to 90 m.
- Daily crossings (ingoing or outgoing)
  - 22 HSC/Catamaran
  - 8-10 Hydrofoil
- A natural laboratory for the study of long waves.
  - Also relevant for tsunami research.

Summary

- Long wave equations are useful for numerical simulations.
- Several different long wave equations formulated for computation of ship waves.
  - KdV and KP equations preferable for analysis
  - Boussinesq equations preferable for numerical simulations
- The COULWAVE model solves Boussineq-type equations, and includes several useful features:
  - Open boundary conditions
  - Wetting/drying in coastal zone
  - Wave breaking model
  - Bottom friction models
- COULWAVE model can simulate long wave phenomena on small and large scales: tsunamies, storm surges, ship waves